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1972 J. Phys. A: Gen. Phys. 5 1037

(http://iopscience.iop.org/0022-3689/5/7/013)

J. Phys. A: Gen. Phys., Vol. 5, July 1972. Printed in Great Britain. © 1972.

Some remarks on an equivalence theorem for an interacting massive spin one particle in quantum field theory

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MS received 1 March 1972

Abstract. The quantization of a massive spin one particle field, satisfying the first order system of equations due to Proca, is summarized. The interaction of such a field with an arbitrary field is then considered. Equivalent theqries in terms of vector and antisymmetric second rank tensor spin one fields are then constructed in a simple manner, providing a generalization of an equivalence theorem due to the author. In addition, this approach, involving the use of first order systems of equations, allows a simple extension to the case of equivalence theorems for arbitrary integral spin. The nature of this extension is indicated.

1. Introduction

In an earlier paper (Jenkins 1972), it was shown, by means of a simple example, how equivalent theories of a Dirac field interacting with a massive spin one field, which transforms under either the $(\frac{1}{2},\frac{1}{2})$ or the $(1,0)\oplus(0,1)$ representation ($SU(2)\otimes SU(2)$ decomposition) of the Lorentz group, may be constructed. In that paper, the transformation connecting the corresponding Lagrangians and equations of motion was established by way of a rather artificial intermediate theory. In the present paper, this unsatisfactory feature is eliminated with the aid of the Proca formalism for spin one (Proca 1936), which involves describing a spin one particle by a field transforming under the $(\frac{1}{2},\frac{1}{2})\oplus(1,0)\oplus(0,1)$ representation of the Lorentz group. The advantages of this approach stem mainly from the natural way in which Proca's formalism falls between the above spin one formalisms.

The plan of the paper is as follows. In § 2 the quantization of the Proca system is summarized, and the Lagrangian and equations of motion governing its interaction with an arbitrary field are set up. The equivalent theories in terms of $(\frac{1}{2},\frac{1}{2})$ and $(1,0)\oplus(0,1)$ spin one fields are then constructed in § 3. In § 4 a generalization of the approach, to include the case of arbitrary integral spin, is indicated. Finally, a discussion of the work is given in § 5.

2. The Proca system†

For simplicity, the $(\frac{1}{2},\frac{1}{2})\oplus(1,0)\oplus(0,1)$ Proca field is taken to be hermitian

$$\chi^{\dagger}(x) = \chi(x) = \begin{pmatrix} V_{\mu}(x) \\ \phi_{\alpha\beta}(x) \end{pmatrix}$$

† Throughout this paper, all the considerations are to be understood as being in terms of the Heisenberg picture.

and its free Lagrangian is given by

$$\mathscr{L}_\chi(x) = \frac{\mu}{2}[V^\mu(x)\phi^{\alpha\beta}(x)]\begin{pmatrix} \mu g_{\mu\nu} & \frac{1}{\sqrt{2}}(g_{\rho\mu}\partial_\lambda - g_{\lambda\mu}\partial_\rho) \\ \frac{1}{\sqrt{2}}(g_{\beta\nu}\overleftarrow{\partial}_\alpha - g_{\alpha\nu}\overleftarrow{\partial}_\beta) & \mu l_{\alpha\beta\lambda\rho} \end{pmatrix}\begin{pmatrix} V^\nu(x) \\ \phi^{\lambda\rho}(x) \end{pmatrix} \tag{1}$$

where the notation here, and throughout this paper is that of Jenkins (1972). On discarding the appropriate four-divergence terms, the Lagrangian (1) may be written equivalently as

$$\mathscr{L}_\chi(x) = \frac{\mu}{2}[V^\mu(x)\phi^{\alpha\beta}(x)]\begin{pmatrix} \mu g_{\mu\nu} & -\frac{1}{\sqrt{2}}(g_{\rho\mu}\overleftarrow{\partial}_\lambda - g_{\lambda\mu}\overleftarrow{\partial}_\rho) \\ -\frac{1}{\sqrt{2}}(g_{\beta\nu}\partial_\alpha - g_{\alpha\nu}\partial_\beta) & \mu l_{\alpha\beta\lambda\rho} \end{pmatrix}\begin{pmatrix} V^\nu(x) \\ \phi^{\lambda\rho}(x) \end{pmatrix} \tag{2}$$

The Lagrangian (1) will be most convenient for deducing the field equations and Lagrangian for the antisymmetric tensor $\phi_{\alpha\beta}(x)$ alone, whilst (2) will be most convenient for a consideration of the vector field $V_\mu(x)$.

In both the cases (1) and (2), the resulting equations of motion are

$$\Lambda(\partial)\chi(x) \equiv \begin{pmatrix} \mu^2 g_{\mu\nu} & \frac{\mu}{\sqrt{2}}(g_{\rho\mu}\partial_\lambda - g_{\lambda\mu}\partial_\rho) \\ -\frac{\mu}{\sqrt{2}}(g_{\beta\nu}\partial_\alpha - g_{\alpha\nu}\partial_\beta) & \mu^2 l_{\alpha\beta\lambda\rho} \end{pmatrix}\begin{pmatrix} V^\nu(x) \\ \phi^{\lambda\rho}(x) \end{pmatrix} = 0. \tag{3}$$

The field $\chi(x)$ may now be quantized, and, following Takahashi (1969), the Klein–Gordon divisor $d(\partial)$ is defined by

$$\Lambda(\partial)d(\partial) = d(\partial)\Lambda(\partial) = (\partial^2 + \mu^2)l$$

where

$$l = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & l_{\alpha\beta\lambda\rho} \end{pmatrix}$$

whence its form is readily found to be

$$d(\partial) = \begin{pmatrix} g_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{\mu^2} & -\frac{1}{\mu\sqrt{2}}(g_{\rho\mu}\partial_\lambda - g_{\lambda\mu}\partial_\rho) \\ \frac{1}{\mu\sqrt{2}}(g_{\beta\mu}\partial_\alpha - g_{\alpha\mu}\partial_\beta) & \frac{1}{\mu^2}((\partial^2 + \mu^2)l_{\alpha\beta\lambda\rho} - A_{\alpha\beta\lambda\rho}(\partial)) \end{pmatrix}. \tag{4}$$

The commutator and free-particle propagator of $\chi(x)$ are now given in terms of the Klein–Gordon divisor as follows:

$$[\chi(x), \tilde{\chi}(x')] = -\mathrm{i}d(\partial)\Delta(x-x') \tag{5}$$

and

$$\langle T(\chi(x), \tilde{\chi}(x'))\rangle_0 = -\mathrm{i}d(\partial)\Delta_c(x-x') - \mathrm{i}[\theta(x_0 - x'_0), d(\partial)]\Delta(x-x') \tag{6}$$

where

$$\tilde{\chi}(x) = [V_\mu(x)\phi_{\alpha\beta}(x)].$$

It should be remarked that the free Proca field equations, (3), are just the transformation equations which connect the free $(\frac{1}{2}, \frac{1}{2})$ and $(1,0)\oplus(0,1)$ formalisms. As will be seen below, this property of the Proca formalism remains even in the presence of interaction.

The Proca field $\chi(x)$ is now assumed to interact with a field $\psi(x)$ described by the free Lagrangian $\mathscr{L}_\psi(x)$, and the consequent field equations

$$\pi(\partial)\psi(x) = 0. \tag{7}$$

The Lagrangian for the interacting system is written in the general form

$$\mathscr{L}(x) = \mathscr{L}_\psi(x) + \mathscr{L}_\chi(x) - J - \tfrac{1}{2}\left[K_\mu \frac{1}{\sqrt{2}} L_{\alpha\beta}\right]\begin{pmatrix} V^\mu(x) \\ \phi^{\alpha\beta}(x)\end{pmatrix}$$
$$-\tfrac{1}{2}[V^\mu(x)\phi^{\alpha\beta}(x)]\begin{pmatrix} K_\mu \\ \frac{1}{\sqrt{2}} L_{\alpha\beta}\end{pmatrix} \tag{8}$$

where J, K_μ and $L_{\alpha\beta}$ are, initially, assumed to be functionals of $\psi(x)$ and its conjugate field, $\bar{\psi}(x)$, only, some more general remarks being reserved for the discussion of § 5; and where $\mathscr{L}_\chi(x)$ is understood as being given by either of the equivalent forms (1) and (2), whichever is the more convenient. The equations of motion consequent to (8) are

$$\pi(\partial)\psi(x) = \frac{\partial J}{\partial\bar{\psi}(x)} + \tfrac{1}{2}\left\{\frac{\partial K^\mu}{\partial\bar{\psi}(x)}, V_\mu(x)\right\} + \frac{1}{2\sqrt{2}}\left\{\frac{\partial L^{\alpha\beta}}{\partial\bar{\psi}(x)}, \phi_{\alpha\beta}(x)\right\} \tag{9}$$

$$\mu^2 V_\mu(x) + \frac{\mu}{\sqrt{2}}(\partial^\lambda\phi_{\lambda\mu}(x) - \partial^\lambda\phi_{\mu\lambda}(x)) = K_\mu \tag{10}$$

$$\mu^2\phi_{\mu\nu}(x) - \frac{\mu}{\sqrt{2}}(\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x)) = \frac{1}{\sqrt{2}} L_{\mu\nu} \tag{11}$$

where $\{,\}$ denotes the anticommutator.

3. The equivalence theorem

The theory of a $(1,0)\oplus(0,1)$ spin one field interacting with the field $\psi(x)$, which is equivalent to that given by the Lagrangian (8), is first constructed. To this end, $\mathscr{L}_\chi(x)$ in (8) is assumed to have the form (1), whence, on using the equation of motion (10), $V_\mu(x)$ is eliminated from (8) in a trivial manner. The resulting Lagrangian is

$$\mathscr{L}(x) = \mathscr{L}_\psi(x) - \tfrac{1}{4}(\partial^\lambda\phi_{\lambda\mu}(x) - \partial^\lambda\phi_{\mu\lambda}(x))(\partial_\rho\phi^{\rho\mu}(x) - \partial_\rho\phi^{\mu\rho}(x))$$
$$+ \tfrac{1}{2}\mu^2\phi_{\mu\nu}(x)\phi^{\mu\nu}(x) - J + \frac{1}{2\sqrt{2}\mu}\{K^\mu, \partial^\lambda\phi_{\lambda\mu}(x) - \partial^\lambda\phi_{\mu\lambda}(x)\}$$
$$-\frac{1}{2\sqrt{2}}\{L_{\mu\nu}, \phi^{\mu\nu}(x)\} - \frac{1}{2\mu^2}K^\mu K_\mu. \tag{12}$$

The corresponding equations of motion may be obtained either directly from this Lagrangian, or by using (10) and its derivatives to eliminate $V_\mu(x)$ from (9) and (11).

In both cases the field equations are

$$\pi(\partial)\psi(x) = \frac{\partial J}{\partial\bar{\psi}(x)} - \frac{1}{2\sqrt{2}\mu}\left\{\frac{\partial K^\mu}{\partial\bar{\psi}(x)}, \partial^\lambda\phi_{\lambda\mu}(x) - \partial^\lambda\phi_{\mu\lambda}(x)\right\}$$
$$+\frac{1}{2\sqrt{2}}\left\{\frac{\partial L^{\lambda\rho}}{\partial\bar{\psi}(x)}, \phi_{\lambda\rho}(x)\right\} + \frac{1}{2\mu^2}\left\{\frac{\partial K^\mu}{\partial\bar{\psi}(x)}, K_\mu\right\} \tag{13}$$

and

$$\tfrac{1}{2}(\partial_\mu\partial^\lambda\phi_{\lambda\nu}(x) - \partial_\mu\partial^\lambda\phi_{\nu\lambda}(x) + \partial_\nu\partial^\lambda\phi_{\mu\lambda}(x) - \partial_\nu\partial^\lambda\phi_{\lambda\mu}(x)) + \mu^2\phi_{\mu\nu}(x)$$
$$= \frac{1}{\sqrt{2}}L_{\mu\nu} + \frac{1}{\sqrt{2}\mu}(\partial_\mu K_\nu - \partial_\nu K_\mu). \tag{14}$$

The inverse transformation is effected by taking (10) as defining the vector part $V_\mu(x)$ of the Proca field $\chi(x)$ in terms of the antisymmetric tensor $\phi_{\alpha\beta}(x)$, and reversing the above procedure. Thus the equivalence of the theories given by the Lagrangians (8) and (12) is established.

Next the equivalent theory in terms of the $(\frac{1}{2}, \frac{1}{2})$ spin one field is constructed. This time, $\mathscr{L}_\chi(x)$ in (8) is assumed to be of the form (2), whence, on using the equation of motion (11), $\phi_{\alpha\beta}(x)$ is eliminated from (8) in a trivial manner. The resulting Lagrangian is

$$\mathscr{L}(x) = \mathscr{L}_\psi(x) - \tfrac{1}{4}(\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x))(\partial^\mu V^\nu(x) - \partial^\nu V^\mu(x))$$
$$+\tfrac{1}{2}\mu^2 V_\mu(x)V^\mu(x) - J - \tfrac{1}{2}\{K^\mu, V_\mu(x)\}$$
$$-\frac{1}{4\mu}\{L_{\mu\nu}, \partial^\mu V^\nu(x) - \partial^\nu V^\mu(x)\} - \frac{1}{4\mu^2}L_{\mu\nu}L^{\mu\nu}. \tag{15}$$

Again the equations of motion may either be obtained directly from this Lagrangian, or by using (11) and its derivatives to eliminate $\phi_{\alpha\beta}(x)$ from (9) and (10). In both cases the field equations are

$$\pi(\partial)\psi(x) = \frac{\partial J}{\partial\bar{\psi}(x)} + \tfrac{1}{2}\left\{\frac{\partial K^\mu}{\partial\bar{\psi}(x)}, V_\mu(x)\right\} + \frac{1}{4\mu}\left\{\frac{\partial L^{\lambda\rho}}{\partial\bar{\psi}(x)}, \partial_\lambda V_\rho(x) - \partial_\rho V_\lambda(x)\right\}$$
$$+\frac{1}{4\mu^2}\left\{\frac{\partial L^{\lambda\rho}}{\partial\bar{\psi}(x)}, L_{\lambda\rho}\right\} \tag{16}$$

and

$$\partial^\mu(\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x)) + \mu^2 V_\nu(x) = K_\nu - \frac{1}{\mu}\partial^\mu L_{\mu\nu}. \tag{17}$$

The inverse transformation is effected by taking (11) as defining the antisymmetric tensor part $\phi_{\alpha\beta}(x)$ of the Proca field $\chi(x)$ in terms of the vector $V_\mu(x)$, and reversing the above procedure. Thus the equivalence of the theories given by the Lagrangians (8) and (15) is established.

This result, in conjunction with the preceding result, establishes the equivalence of the three theories given by the Lagrangians (8), (12) and (15).

4. Arbitrary integral spin

The extension of the above approach for spin one, to include the arbitrary integral spin case, is prefixed by two remarks on the former, which indicate the method and scope of the extension.

Firstly, it is noted that, when the Lorentz group is extended to include parity, the only irreducible representations, which contain spin one as the maximum spin, are $(\frac{1}{2}, \frac{1}{2})$ and $(1, 0)\oplus(0, 1)$. Secondly, as seen above, spin one fields transforming under these representations are naturally connected through the first order Proca formalism.

Now in the case of arbitrary integral spin σ there are $\sigma+1$ irreducible representations of the extended Lorentz group which contain σ as the maximum spin, namely

$$\left(\frac{\sigma}{2}, \frac{\sigma}{2}\right), \left(\frac{\sigma+1}{2}, \frac{\sigma-1}{2}\right)\oplus\left(\frac{\sigma-1}{2}, \frac{\sigma+1}{2}\right), \left(\frac{\sigma+2}{2}, \frac{\sigma-2}{2}\right)\oplus\left(\frac{\sigma-2}{2}, \frac{\sigma+2}{2}\right), \ldots, (\sigma, 0)\oplus(0, \sigma).$$

In addition, any two spin σ fields, which transform under representations, adjacent in the above sequence, are naturally connected through first order equations formally identical with the Proca system (Corson 1953). It is this last point which allows the considerations of §§ 2 and 3 to be extended, almost without change, to the case of arbitrary integral spin.

Let spin σ fields transforming under the above representations be denoted respectively by

$$G^{(0)}_{\mu_1\ldots\mu_\sigma}(x),\ G^{(1)}_{[\mu_1\nu_1]\mu_2\ldots\mu_\sigma}(x),\ G^{(2)}_{[\mu_1\nu_1][\mu_2\nu_2]\mu_3\ldots\mu_\sigma}(x),\ \ldots,\ G^{(\sigma)}_{[\mu_1\nu_1]\ldots[\mu_\sigma\nu_\sigma]}(x)$$

where square brackets denote that the field is antisymmetric under the interchange of the indices within the brackets, whilst all the fields are assumed symmetric and traceless in all the indices not within brackets. Corson (1953) then gives the first order systems of equations, connecting these fields, in the free case, as

$$\partial_\rho G^{(0)}_{\mu_1\mu_2\ldots\mu_\sigma}(x) - \partial_{\mu_1} G^{(0)}_{\rho\mu_2\ldots\mu_\sigma}(x) = \kappa G^{(1)}_{[\rho\mu_1]\mu_2\ldots\mu_\sigma}(x) \tag{18}$$

$$\partial^\rho G^{(1)}_{[\mu_1\rho]\mu_2\ldots\mu_\sigma}(x) - \partial^\rho G^{(1)}_{[\rho\mu_1]\mu_2\ldots\mu_\sigma}(x) = \kappa G^{(0)}_{\mu_1\ldots\mu_\sigma}(x) \tag{19}$$

with formally similar equations satisfied by the pairs of fields $G^{(1)}$ and $G^{(2)}$ etc, and where κ is proportional to the (nonzero) mass of the spin σ particle. The formal similarity of (18) and (19) to the Proca system (3) is evident.

On the basis of this formal similarity, the considerations of §§ 2 and 3 carry over, without any formal change, to any given pair of the equations of which (18) and (19) are an example. Then, given a theory in terms of one of the above spin σ fields, a step by step elimination of the appropriate fields from their first order systems, along the lines of § 3, allows a construction of the equivalent theory in terms of any other of these fields. Because of the demonstrated formal similarity to the spin one case, and the cumbersome nature of the proof, due to the proliferation of first order Proca-like systems of equations, the details of the arbitrary integral spin case are omitted.

5. Discussion

It has been seen that the first order Proca formalism provides a spin one theory, naturally intermediate between the vector and antisymmetric tensor spin one theories. In the free-particle case, and in the presence of an interaction, where the currents, to which the spin one fields couple, depend only on $\psi(x)$ and $\bar{\psi}(x)$, the Proca field equations, given respectively by (3) and (10), (11), just provide the transformations connecting equivalent theories in terms of the vector and antisymmetric tensor spin one fields, which, in turn, are equivalent to the original theory in terms of the Proca field. It is exactly this property of the Proca formalism which allows such a simple and transparent proof of the equivalence theorem in § 3.

As noted by Jenkins (1972) the essential *formal* differences between equivalent theories, of the type considered in that paper and here, lie in the presence of contact terms in the Lagrangian (eg the term $-(1/2\mu^2)K^{\mu}K_{\mu}$ in (12)). As shown in the former work, by a perturbation expansion of the S matrix in the interaction picture, the role of such terms is to compensate for differences in the free-particle propagators of corresponding fields in the equivalent theories. If now the results of § 3 are generalized to allow K_{μ} and $L_{\alpha\beta}$ to be, in addition, functionals of $V_{\mu}(x)$ and $\phi_{\alpha\beta}(x)$, then the above remarks, although still valid, take on a more complicated structure. For, in this case, a compensating contact term will, in general, contain the spin one field, whose presence must be compensated for by a further contact term, and so on. Thus the introduction of these contact terms will be an iterative procedure, in general not ending after a finite number of steps. In terms of the first order Proca formalism, this phenomenon is reflected in the use of the analogues of (10) and (11), to eliminate $V_{\mu}(x)$ or $\phi_{\alpha\beta}(x)$ from the Lagrangian being, also, an iterative procedure.

Finally, the only types of nonderivative interaction, other than mass terms and that considered in § 3, for which the above process of elimination is always finite, are the following:

(i) $K_{\mu} = K_{\mu}(\psi(x), \bar{\psi}(x), \phi_{\alpha\beta}(x))$ $\qquad L_{\alpha\beta} = L_{\alpha\beta}(\psi(x), \bar{\psi}(x))$

(ii) $K_{\mu} = K_{\mu}(\psi(x), \bar{\psi}(x))$ $\qquad L_{\alpha\beta} = L_{\alpha\beta}(\psi(x), \bar{\psi}(x), V_{\mu}(x))$

the dependences on $V_{\mu}(x)$ and $\phi_{\alpha\beta}(x)$ both being linear. An example of these more general cases is provided by the minimal electromagnetic interaction of spin one particles, and this subject is treated in detail in a separate paper.

References

Corson E M 1953 *Introduction to Tensors, Spinors and Relativistic Wave-Equations* (London: Blackie) pp 118–21
Jenkins J D 1972 *J. Phys.* A: *Gen. Phys.* **5** 705–16
Proca A 1936 *J. Phys. Radium* **7** 347–53
Takahashi Y 1969 *An Introduction to Field Quantization* (Oxford: Pergamon)